2.9. Linear Approximation.

Key point: Linearization of f at a = Tangent line of f at a.

Formula: $L(x) = f(a) + f'(a)\cdot(x-a)$. $(*)$.

Goal/Motivation: How to estimate $\sqrt{9.01}$ without calculator? More precisely, intuitively, $\sqrt{9.01} \approx \sqrt{9} = 3$, we want to get a more accurate approximation via $L(x)$.

Method: Pick the suitable function $f(x)$ and a, Apply Linearization Formula $L(x)$ as an approximation of the desired value of $f(x)$.

e.g.1. Consider $f(x) = \sqrt{x}$. Find its Linearization at $x=9$.

Rmk: It is equivalent to ask "Find the tangent line at $x=9$".

Solution: Apply formula $(*)$ directly.

$a=9$, $f(a)=\sqrt{9}=3$, $f'(x)=\frac{1}{2}\cdot x^{\frac{1}{2}-1} = \frac{1}{2}\cdot x^{-\frac{1}{2}} \Rightarrow \boxed{f'(9) = \frac{1}{2}\cdot 9^{-\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{\sqrt{9}} = \frac{1}{6}}$

therefore, Linearization of $f(x)$ at 9 is:

$$\boxed{L(x) = 3 + \frac{1}{6}\cdot(x-9)}.$$

eg2. Use a linearization to find a good approximation of $\sqrt{9.01}$.

(F16). Hint: Consider the function $f(x)=\sqrt{x}$ and point $x=9$ in eg.1.

eg.1 $\Rightarrow$ $L(x) = 3 + \frac{1}{6}\cdot(x-9)$ (Linearization)

Plug in 9.01 (since we want to estimate $\sqrt{9.01}$)

$$L(9.01) = 3 + \frac{1}{6}(9.01-9) = 3 + \frac{1}{6}\cdot 0.01 = \boxed{3 + \frac{1}{600}}$$

eg.3 If $f(1)=3$ and $f'(1)=5$. use linear approximation to estimate $f(0.99)$
(S16, MC). Solution: $a=1$, $f(1)=3$, $f'(1)=5$.

$\Rightarrow$ Linearization: $L(x) = f(1) + f'(1)\cdot(x-1) = 3+5\cdot(x-1)$.

To estimate $f(0.99)$, plug in $x=0.99$, $L(0.99) = 3+5\cdot(0.99-1) = 3+5\cdot(-0.01)$
$= 3-0.05 = \boxed{2.95}$

- Error in the linear approximation.

$L(x) - f(a) = f'(a)\cdot(x-a)$, i.e., Derivative $f'(a)$ times the error of ~~the~~ variable

eg 4. We want to measure the radius of a sphere to calculate its surface area (Formula: $A = 4\pi\cdot r^2$). The radius is measured to be 5 cm with possible error of 0.1 cm. What's the maximum error in the calculated surface area?

Solution: $A(r) = 4\pi\cdot r^2$. $a=5$. $A'(r) = 4\pi\cdot 2r$

(Caution: this is not Related Rates Prob. The derivative is w.r.t. r)

$A(5) = 4\pi\cdot 25 = 100\pi$, $A'(5) = 4\pi\cdot 2\cdot 5 = 40\pi$.

The linearization of $A(r)$ at 5 is: $100\pi + 40\pi\cdot(r-5)$

The error in the estimate is

$f'(a)(x-a)$, i.e., $40\pi\cdot(r-5)$.

Now the error for radius is at most 0.1, i.e., $r-5$ is at most 0.1

Therefore, the error for the surface area is $40\pi\cdot 0.1 = 4\pi$

§3.1 Extreme Values

Key points: ① Absolute (Global) maximum/minimum.

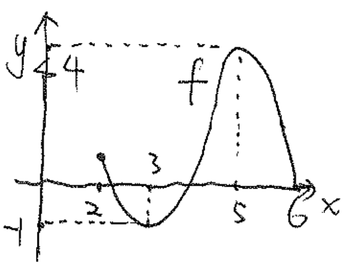
② Critical point (numbers)

③ Local maximum/minimum. (More details in 3.3, 3.5)

Goal: want to find the maximum/minimum values of $f(x)$ through its derivative $f'(x)$.

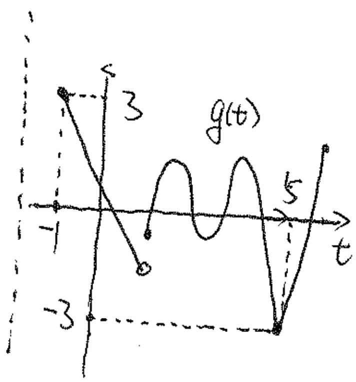
• Def: For $f(x)$ defined on $[a,b]$. If there is c in $[a,b]$ such that $f(c) \geq f(x)$ for all x in $[a,b]$, then we say $f(x)$ has ABSOLUTE MAXIMUM VALUE $f(c)$ at $x=c$. (Or, the absolute maximum occurs at $x=c$.) Absolute minimum is defined similarly with $f(c) \leq f(x)$.

graph:



$f(x)$ on $[2,6]$.

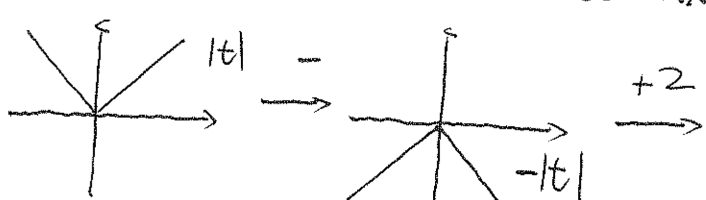
$f(x)$ attains its absolute maximum $f(5)=4$ at 5. and its absolute minimum $f(3)=-1$ at 3.

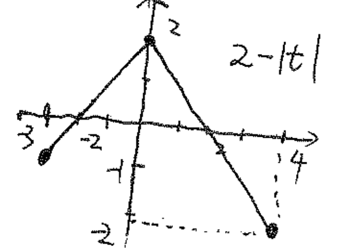


$g(t)$ attains its absolute maximum $g(-1)=3$ at $t=-1$ and its absolute minimum $g(5)=-3$ at $t=5$.

e.g.1. $h(t) = 2-|t|$ on $[-3,4]$. Find its absolute maximum/minimum.

Solution: Draw the graph of $2-|t|$.





$h(t)$: absolute maximum of h is 2, occurs at $t=0$

absolute minimum of h is -2, occurs at $t=4$

• Def: Critical points (numbers) are all the points x such that $f'(x)=0$ or $f'(x)$ is not defined ($f'(x)$ D.N.E / $f(x)$ is not differentiable).

- Method to find all critical points for $f(x)$ on $[a,b]$.

Step 1: Take derivative. (Compute $f'(x)$).

Step 2: Set the denominator of $f'(x)$ to be zero and solve for x. (where $f'(x)$ is undefined).

Step 3: Set $f'(x)=0$. Solve for x. (where $f'(x)$ is zero)

Step 4: Discard those points not in $[a,b]$. All the rest points in Step 2, 3 are your answers. (critical points)

e.g. 2. Let $f(t)=-t+4\sqrt{t}$. Find all critical numbers of f in $[0,25]$

Solution: S1: $f'(t)=(-t+4\sqrt{t})'=(-t)'+(4\cdot t^{\frac{1}{2}})'=-1+4\cdot\frac{1}{2}\cdot t^{-\frac{1}{2}}=\boxed{-1+2t^{-\frac{1}{2}}}$

S2: Notice $f'(t)=-1+2\cdot\frac{1}{\sqrt{t}}$. ($\sqrt{t}$ is in the denominator, $\Rightarrow \sqrt{t}$ can't be zero.

i.e. $\sqrt{t}=0 \Rightarrow \boxed{t=0}$ ($t=0$ is where $f'(t)$ is undefined)

S3: Set $f'(t)=0 \Rightarrow -1+2\cdot\frac{1}{\sqrt{t}}=0 \Rightarrow \frac{2}{\sqrt{t}}=1 \Rightarrow \sqrt{t}=2 \Rightarrow \boxed{t=4}$

S4: Both $0, 4$ are in $[0,25]$. f has critical numbers $t=0, t=4$ in $[0,25]$.

e.g. 3 Find all the critical points for $f(x)=48x-x^3$ over $[-5,2]$

Solution: $f'(x)=(48x-x^3)'=48-3\cdot x^2$.

(No step 2 needed. No denominator in $f'(x)$. i.e. $f'(x)$ is defined for all x).

$f'(x)=0 \Rightarrow 48-3x^2 \Rightarrow 48=3x^2 \Rightarrow x^2=16 \Rightarrow x=\pm 4$.

therefore, f has critical point $\boxed{x=-4 \text{ in } [-5,2]}$. ($x=4$ is not in $[-5,2]$)

Remark: If we change the interval in eg 3 to be $[-1,2]$, then

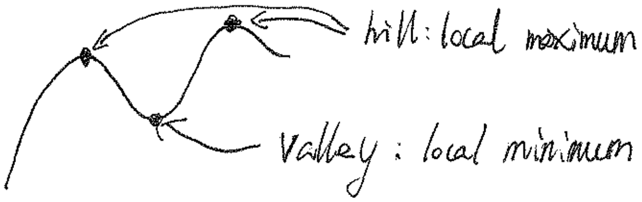
$f(x)=48x-x^3$ has NO CRITICAL POINTS IN $[-1,2]$.

Neither $x=4$ or $x=-4$ is in $[-1,2]$

- Def: For $c \in [a,b]$, if $f(c)$ is the largest (smallest) value for all $f(x)$ near $x=c$, then $f(c)$ is a local maximum (or minimum) of $f(x)$.

Remark: local maximum is a point where the graph has a hill

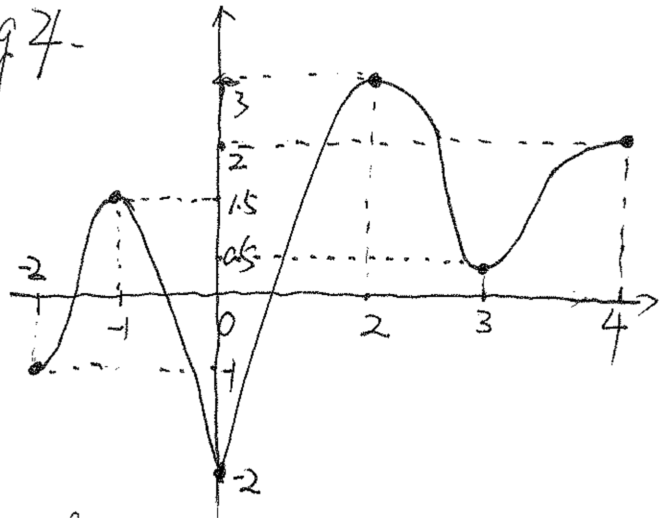
local minimum is a point where the graph has a valley



Absolute maximum is the highest hill.

Absolute ~~valley~~ minimum is the lowest valley.

eg 4.



$f(x)$ on $[-2,4]$

- (Hills) local maximum:

  $f(-1)=1.5$, $f(2)=3$, $f(4)=2$.

- Absolute maximum is the largest among the above, i.e., $f(2)=3$

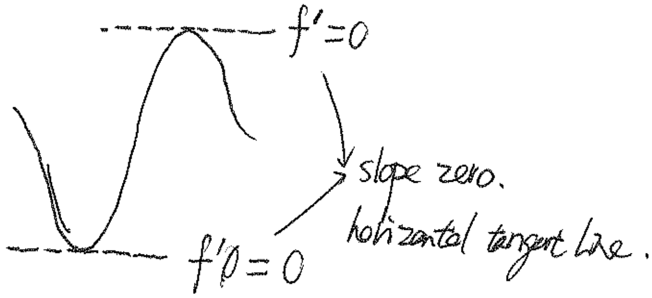
- (Valleys) local minimum:

  $f(-2)=-1$, $f(0)=-2$, $f(3)=0.5$

- Absolute minimum is the smallest among the above, i.e., $f(0)=-2$.

Remark (Theorem): If $f(x)$ has a local extremum (maximum/minimum) at $x=c$ in (a,b), and $f'(c)$ exists, then $f'(c)=0$.

In the graph, it means $f(x)$ has a horizontal tangent line (with slope zero) at these local extremums.



In eg 4. $f'(-1)=0$, $f'(2)=0$, $f'(3)=0$

$f(0)$ is local minimum (and absolute minimum)

but $f'(0)$ does not exist.

☆. Method to find absolute maximum/minimum of $f(x)$ on $[a,b]$

Step 1: Find all CRITICAL POINTS of f in $[a,b]$.

Step 2: List all values of f at the above critical points, $f(c)$.

List $f(a)$, $f(b)$. (the values of f at endpoints)

Step 3: Compare all values listed in Step 2.

The largest is the absolute maximum. The smallest is the absolute minimum.

eg.5. Find the absolute maximum and minimum for $f(t)=-t+4\sqrt{t}$ in $[0,25]$ considered in eg.2.

Solution: Step 1: Critical points: $t=0$, $t=4$.

Step 2: f at critical points: $f(0)=0$, $f(4)=-4+4\cdot\sqrt{4}=4$.

f at ~~critical~~ endpoints: $f(0)=0$, $f(25)=-25+4\cdot\sqrt{25}=-5$

Step 3: $f(0)=0$, $\underbrace{f(4)=4}_{\text{largest}}$, $\underbrace{f(25)=-5}_{\text{smallest}}$.

f has absolute maximum $f(4)=4$, occurs at $t=4$

f has absolute minimum $f(25)=-5$, occurs at $t=25$.

eg 6. Find the absolute maximum value of $f(x)=48x-x^3$ over $[-1,2]$

(S16, MC) $f'(x)=48-3x^2=0 \Rightarrow x^2=16 \Rightarrow x=\pm 4$ not in $[-1,2]$

$f(x)$ has no critical points in $[-1,2]$. (See Remark after eg 3).

(List endpoints only): $x=-1$, $x=2$.
$$f(-1)=48\cdot(-1)-(-1)^3=-48+1=-47$$
$$f(2)=48\cdot 2-2^3=96-8=88$$

Therefore, the absolute maximum of f over $[-1,2]$ is $\boxed{88}$, which occurs at $x=2$.